Algorithm Problem Solving (APS):
Dynamic Programming

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UC San Diego SPIS 2019

Example: The 0-1 Knapsack Problem

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- Imagine you’re in a heist crew en route to your next job
- Your target is a truck carrying electronic goods
- The modified Honda Civic you’re driving has a weight limit
    - You know how much each item costs and weighs
    - Which items do you steal to maximize your profits?

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- **Input:** A list of *n* items, where the *i*-th item has weight w_i and value v_i,

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- **Output:** A set of items such that the sum of their weights is below x

  and the sum of their values is **maximized**

Example: The 0-1 Knapsack Problem

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$x = 20$

Ex

Can we use a Greedy Algorithm?

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$x = 20$

Nope! Not guaranteed to be optimal!

Simpler Example: Fibonacci Sequence

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- The first number is **0**

0

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- The first number is 0
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0, **1**

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0, 1, 1, 2, 3, 5, 8, 13, ...

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Simpler Example: Fibonacci Sequence

Write a recursive Python function!

- **Output:** The ***n***-th number of the Fibonacci sequence (0-based indexing)

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Recursive Fibonacci

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def fib_rec(n):
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Yay! 😊

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Hold up... 🤔

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def fib_rec(n):
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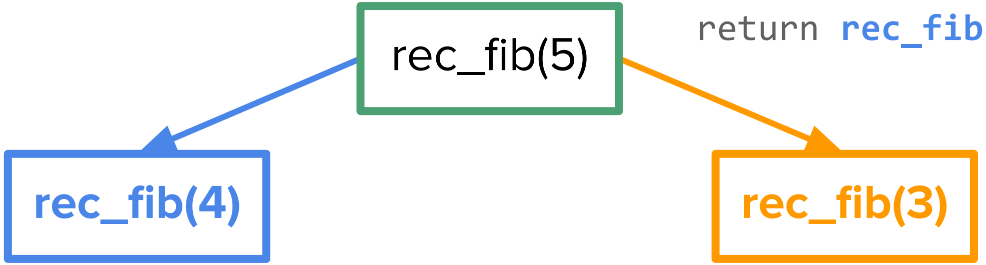

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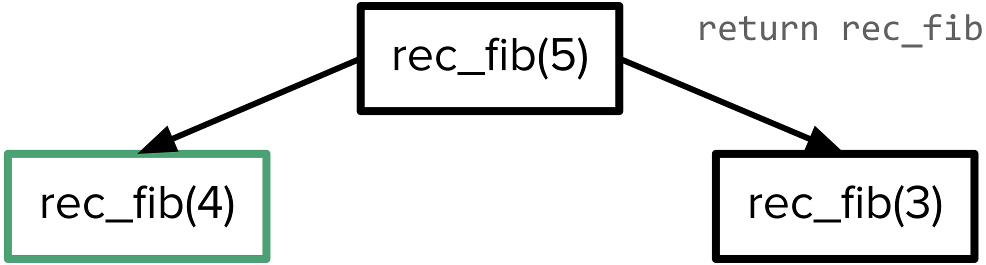

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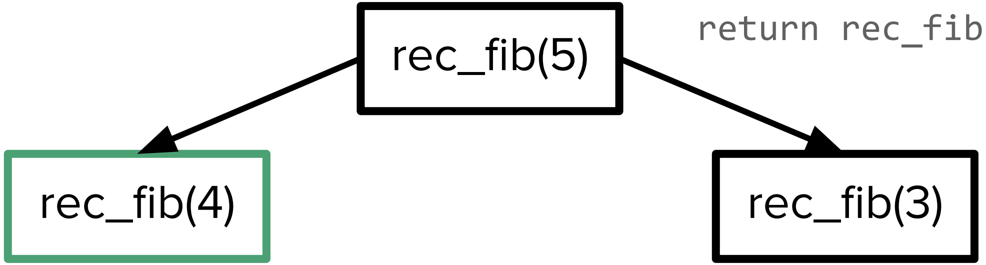

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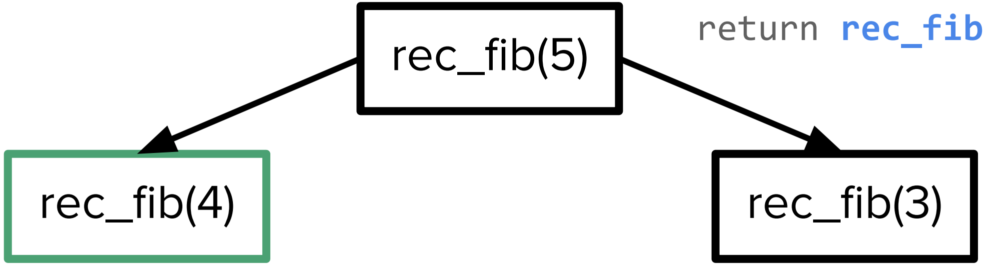

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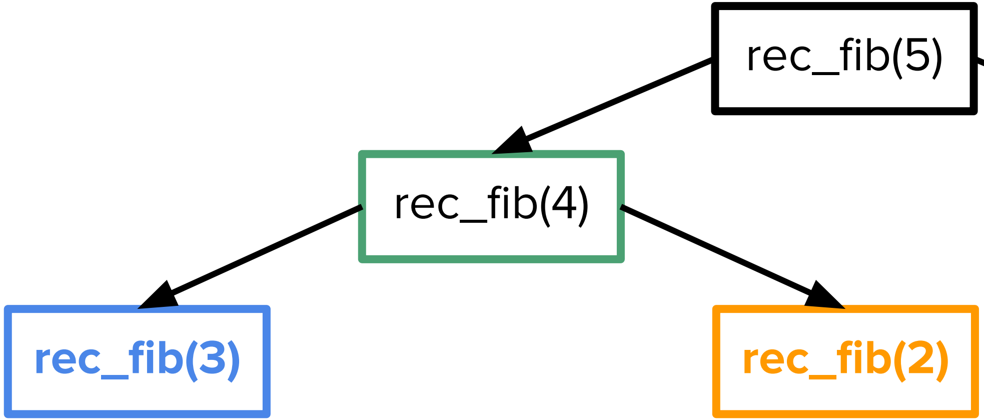

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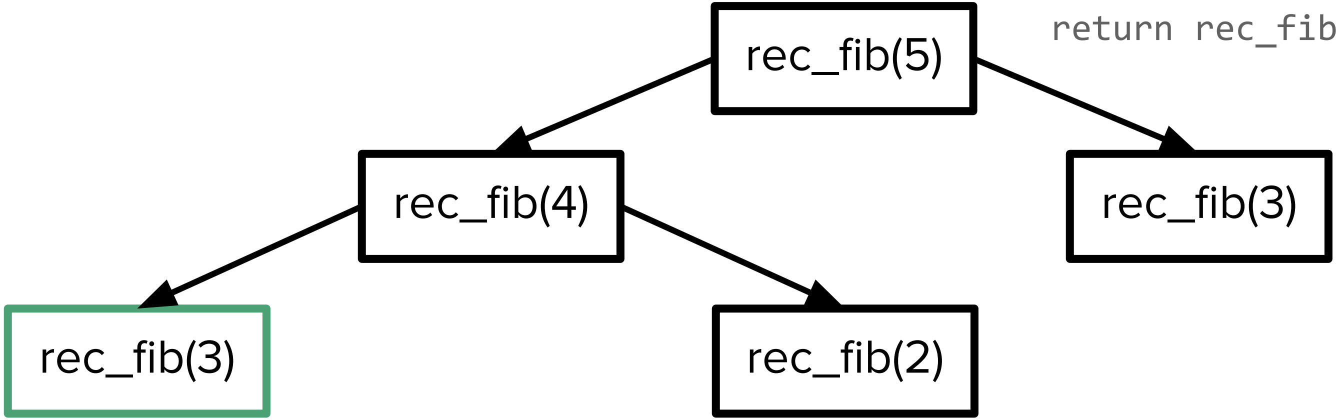

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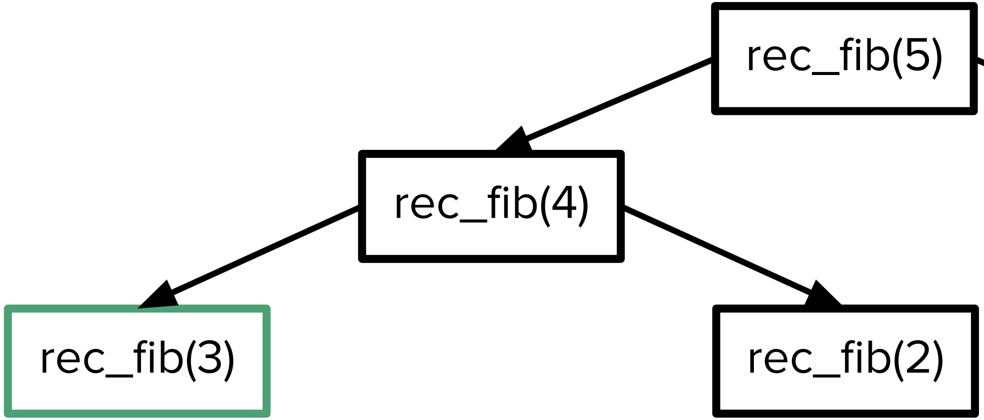

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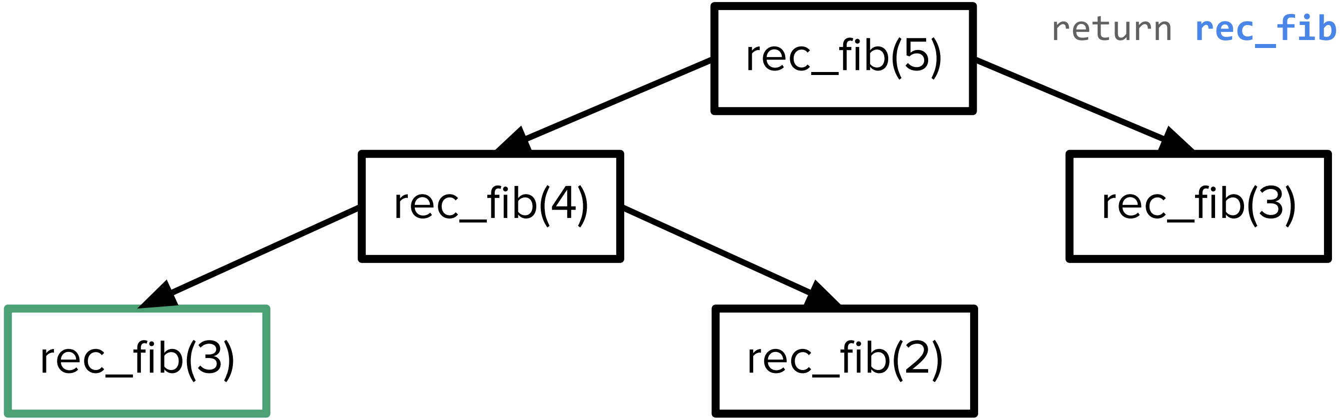

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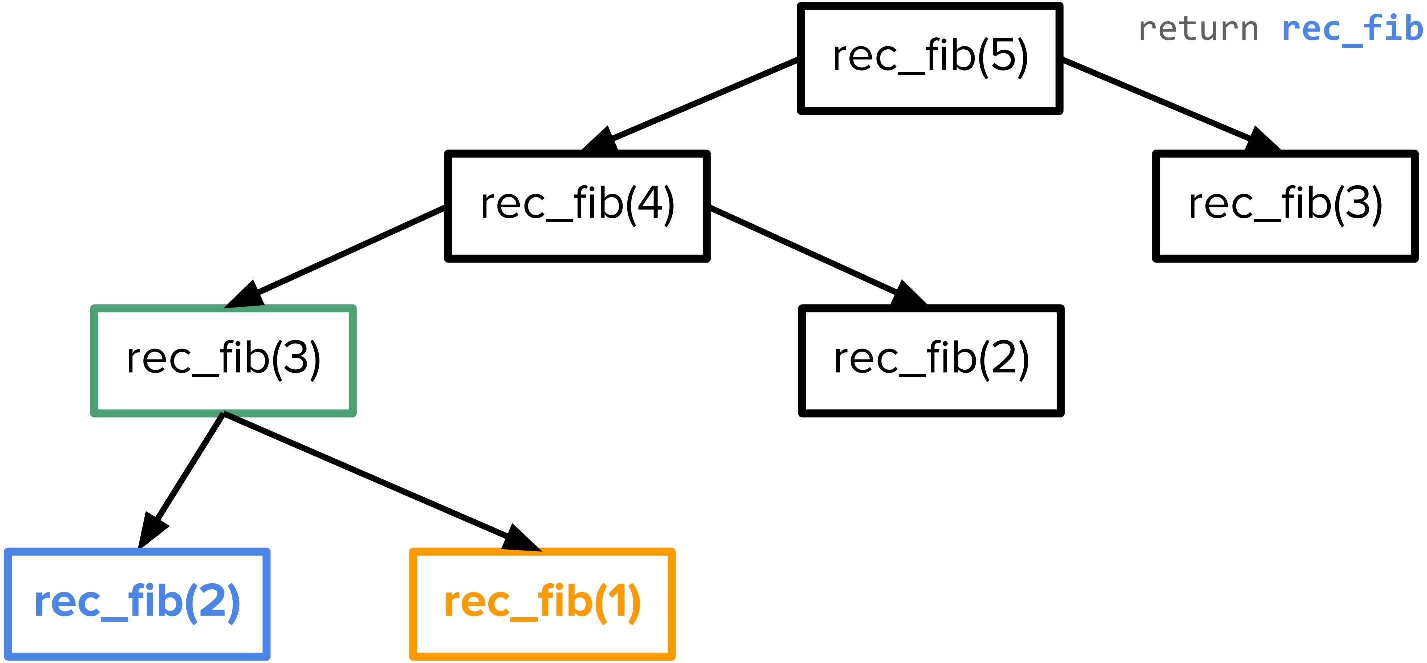

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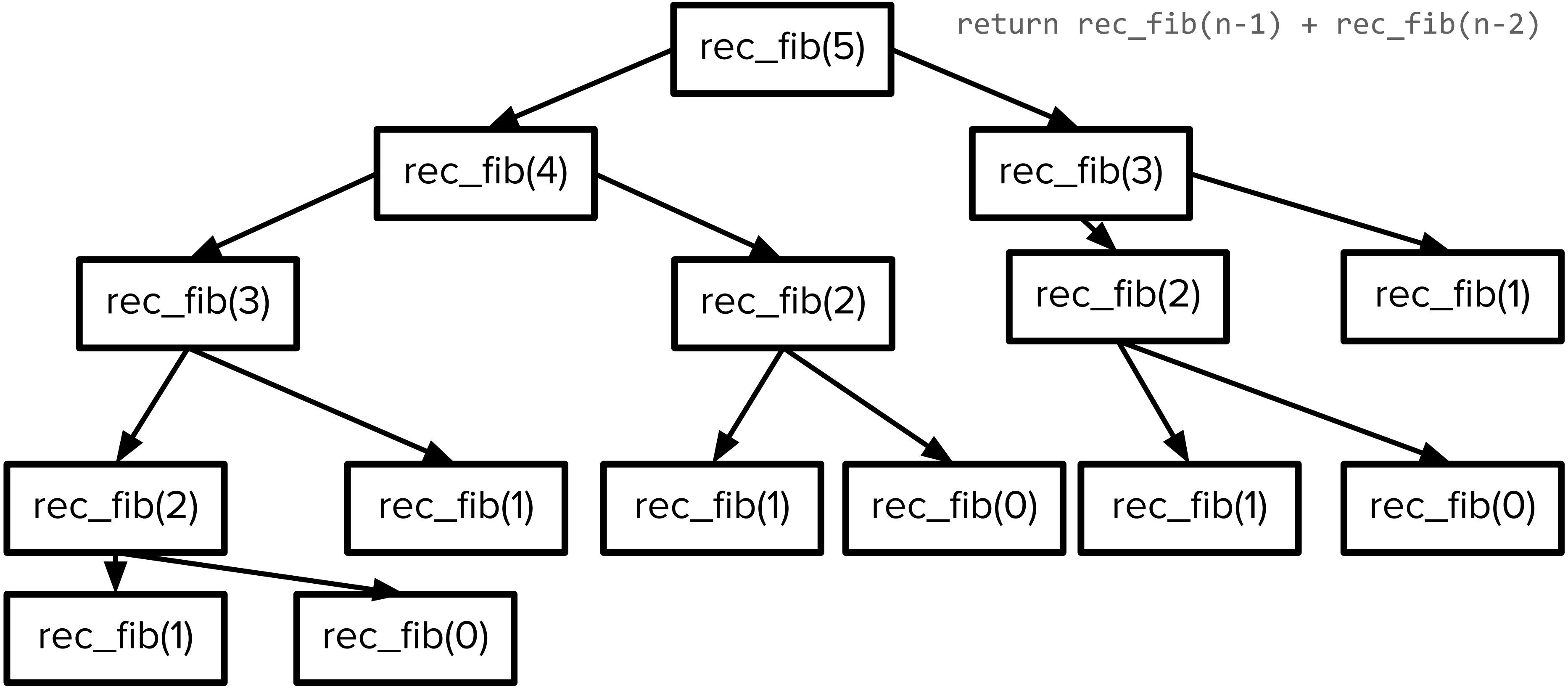

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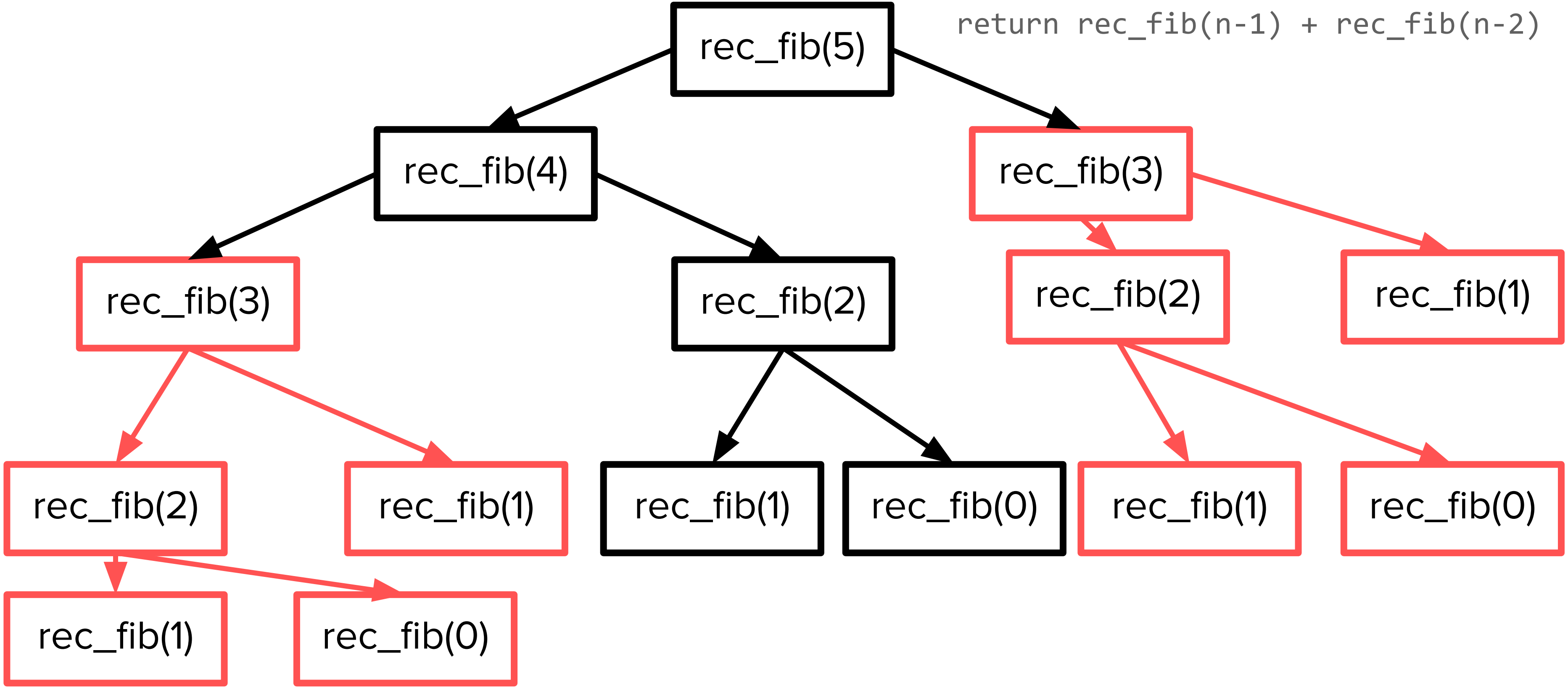

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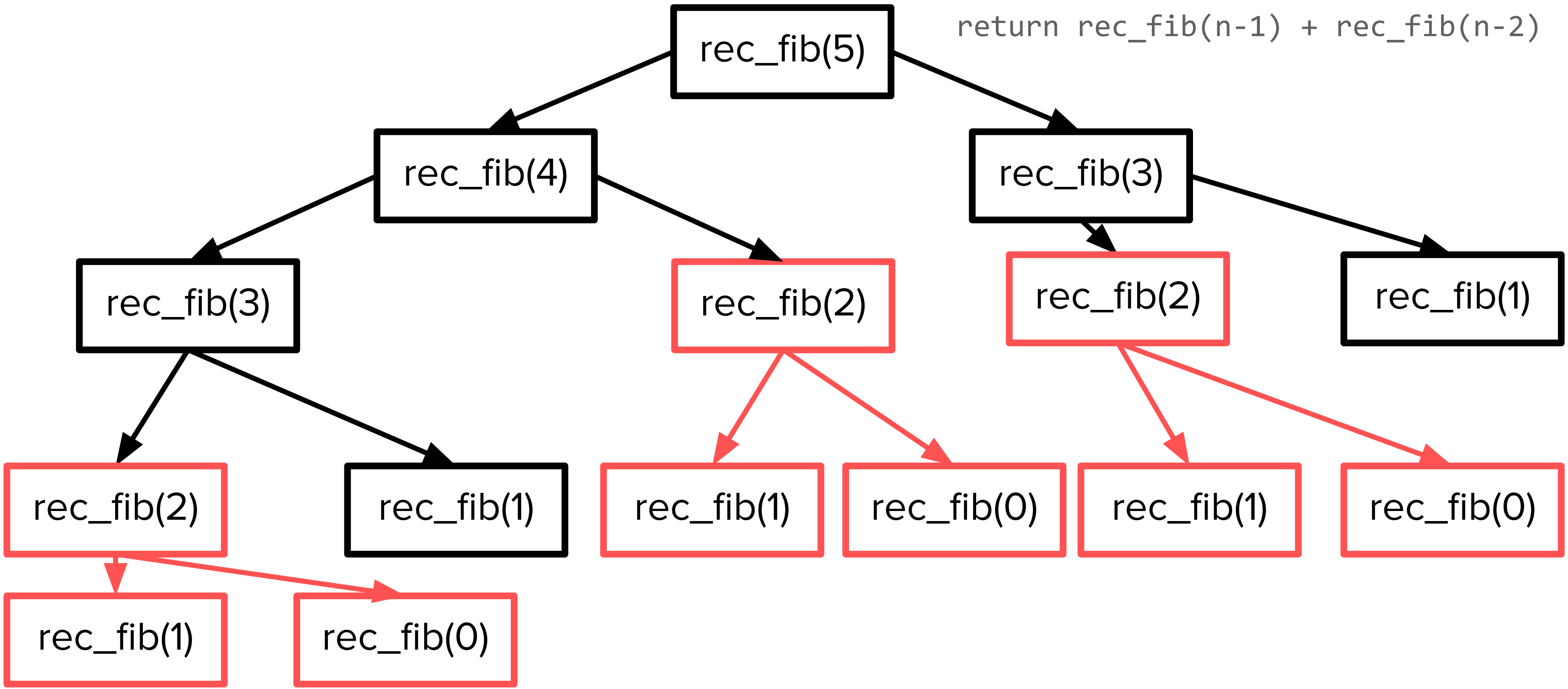

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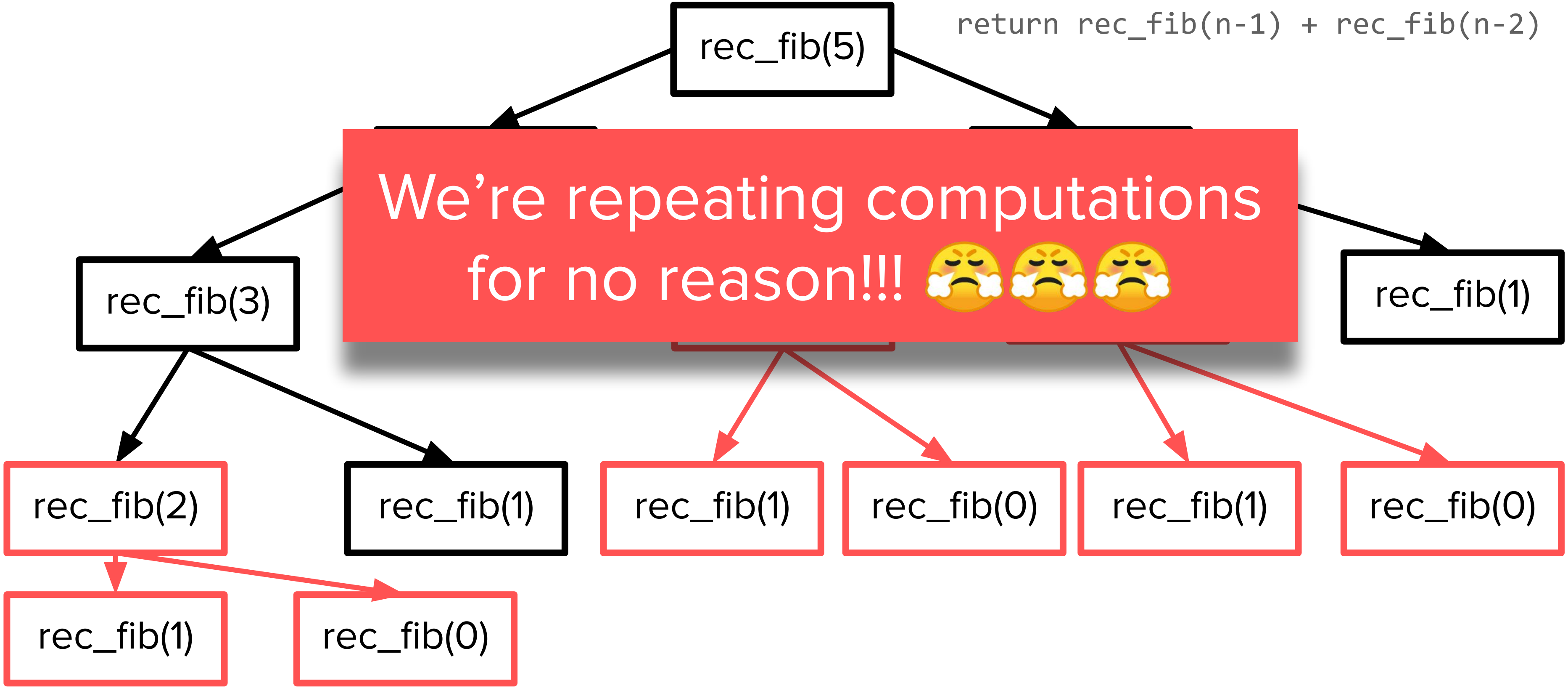

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  - The values of rec_fib(3) and rec_fib(2) never change, so once we compute them the first time, why not just save them somewhere?
- **Memoization:** Saving the results of expensive functions somewhere and returning the saved result when the same inputs occur again

Fast Fib(e)

fast_fib(5)

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$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{if } n > 1 \end{cases}$$

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n:	0	1	2	3	4	5
fib(*n*):						

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    - No value added, so it's just our best scenario of looking at the first *i*-1 items with the same (*j*) total weight
  - Take the **maximum** of these two possible options

Example: The 0-1 Knapsack Problem

$$P(i, j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1, j)\} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \end{cases}$$

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1, j)\} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \end{cases}$$

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j **(total weight)**

i **(number of items seen)**

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0																					
1																					
2																					
3																					
4																					

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} \boxed{0 \qquad\qquad \text{if } i = 0} \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

***j* (total weight)**

***i* (number of items seen)**

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0																					
1																					
2																					
3																					
4																					

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} \boxed{0 \qquad \text{if } i = 0} \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

j (total weight)

i (number of items seen)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1																					
2																					
3																					
4																					

Example: The 0-1 Knapsack Problem

x = 20

w:	3	8	10	6
v:	$5	$5	$13	$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

***j* (total weight)**

***i* (number of items seen)**

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1																					
2																					
3																					
4																					

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

***j* (total weight)**

***i* (number of items seen)**

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	?																				
2																					
3																					
4																					

$x = 20$

Example: The 0-1 Knapsack Problem

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

***j* (total weight)**

***i* (number of items seen)**

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0										0	0	0	0	0	0
1	?																				
2																					
3																					
4																					

Add: \$5 + -∞

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

***j* (total weight)**

***i* (number of items seen)**

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0									0	0	0	0	0	0
1	?																				
2																					
3																					
4																					

Add: \$5 + -∞

Don't Add: \$0

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

***j* (total weight)**

***i* (number of items seen)**

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0									0	0	0	0	0	0
1	0																				
2																					
3																					
4																					

Add: \$5 + -∞

Don't Add: \$0

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \end{cases}$$

***j* (total weight)**

***i* (number of items seen)**

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0																				
2	0																				
3	0																				
4	0																				

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1, j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

j **(total weight)**

i **(number of items seen)**

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0																			
2	0	0																			
3	0	0																			
4	0	0																			

Example: The 0-1 Knapsack Problem

x = 20

w:	3	8	10	6
v:	$5	$5	$13	$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

j **(total weight)**

i **(number of items seen)**

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0																		
2	0	0	0																		
3	0	0	0																		
4	0	0	0																		

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{\boxed{v_i + P(i-1, j-w_i)}, P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

j (total weight)

i (number of items seen)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5																	
2	0	0	0																		
3	0	0	0																		
4	0	0	0																		

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), \boxed{P(i-1,j)}\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

j (total weight)

i (number of items seen)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5																	
2	0	0	0	5																	
3	0	0	0																		
4	0	0	0																		

$x = 20$

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Example: The 0-1 Knapsack Problem

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

***j* (total weight)**

***i* (number of items seen)**

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5																	
2	0	0	0	5																	
3	0	0	0	5																	
4	0	0	0	5																	

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

***j* (total weight)**

***i* (number of items seen)**

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5																
2	0	0	0	5	5																
3	0	0	0	5	5																
4	0	0	0	5	5																

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1, j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

***j* (total weight)**

***i* (number of items seen)**

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5															
2	0	0	0	5	5	5															
3	0	0	0	5	5	5															
4	0	0	0	5	5	5															

x = 20

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

Example: The 0-1 Knapsack Problem

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1, j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

***j* (total weight)**

i (number of items seen)

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5														
2	0	0	0	5	5	5	5														
3	0	0	0	5	5	5	5														
4	0	0	0	5	5	5															

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

j (total weight)

i (number of items seen)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5														
2	0	0	0	5	5	5	5														
3	0	0	0	5	5	5	5														
4	0	0	0	5	5	5	6														

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

Example: The 0-1 Knapsack Problem

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

j (total weight)

i (number of items seen)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5													
2	0	0	0	5	5	5	5	5													
3	0	0	0	5	5	5	5	5													
4	0	0	0	5	5	5	6	6													

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

j **(total weight)**

i **(number of items seen)**

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5												
2	0	0	0	5	5	5	5	5													
3	0	0	0	5	5	5	5	5													
4	0	0	0	5	5	5	6	6													

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), \boxed{P(i-1,j)}\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

***j* (total weight)**

***i* (number of items seen)**

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5												
2	0	0	0	5	5	5	5	5	5												
3	0	0	0	5	5	5	5	5													
4	0	0	0	5	5	5	6	6													

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

Example: The 0-1 Knapsack Problem

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{\boxed{v_i + P(i-1, j-w_i)}, \boxed{P(i-1,j)}\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

j (total weight)

i (number of items seen)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5												
2	0	0	0	5	5	5	5	5	5												
3	0	0	0	5	5	5	5	5													
4	0	0	0	5	5	5	6	6													

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

Break ties arbitrarily

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

***j* (total weight)**

***i* (number of items seen)**

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5												
2	0	0	0	5	5	5	5	5	5												
3	0	0	0	5	5	5	5	5													
4	0	0	0	5	5	5	6	6													

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

***j* (total weight)**

***i* (number of items seen)**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5												
2	0	0	0	5	5	5	5	5	5												
3	0	0	0	5	5	5	5	5	5												
4	0	0	0	5	5	5	6	6	6												

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

j **(total weight)**

i **(number of items seen)**

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5											
2	0	0	0	5	5	5	5	5	5	5											
3	0	0	0	5	5	5	5	5	5	5											
4	0	0	0	5	5	5	6	6	6												

Example: The 0-1 Knapsack Problem

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

j (total weight)

i (number of items seen)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5											
2	0	0	0	5	5	5	5	5	5	5											
3	0	0	0	5	5	5	5	5	5	5											
4	0	0	0	5	5	5	6	6	6	11											

Example: The 0-1 Knapsack Problem

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***j* (total weight)**

***i* (number of items seen)**

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5										
2	0	0	0	5	5	5	5	5	5	5	5										
3	0	0	0	5	5	5	5	5	5	5											
4	0	0	0	5	5	5	6	6	6	11											

$x = 20$

w:	3	8	10	6
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j **(total weight)**

i **(number of items seen)**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5										
2	0	0	0	5	5	5	5	5	5	5	5										
3	0	0	0	5	5	5	5	5	5	5	13										
4	0	0	0	5	5	5	6	6	6	11											

$x = 20$

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***j* (total weight)**

***i* (number of items seen)**

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5										
2	0	0	0	5	5	5	5	5	5	5	5										
3	0	0	0	5	5	5	5	5	5	5	13										
4	0	0	0	5	5	5	6	6	6	11	13										

Example: The 0-1 Knapsack Problem

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***j* (total weight)**

***i* (number of items seen)**

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5									
2	0	0	0	5	5	5	5	5	5	5	5										
3	0	0	0	5	5	5	5	5	5	5	13										
4	0	0	0	5	5	5	6	6	6	11	13										

$x = 20$

Example: The 0-1 Knapsack Problem

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j (total weight)

i (number of items seen)	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5									
2	0	0	0	5	5	5	5	5	5	5	5	10									
3	0	0	0	5	5	5	5	5	5	5	13										
4	0	0	0	5	5	5	6	6	6	11	13										

Example: The 0-1 Knapsack Problem

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***j* (total weight)**

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	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5									
2	0	0	0	5	5	5	5	5	5	5	5	10									
3	0	0	0	5	5	5	5	5	5	5	13	13									
4	0	0	0	5	5	5	6	6	6	11	13	13									

Example: The 0-1 Knapsack Problem

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	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5								
2	0	0	0	5	5	5	5	5	5	5	5	10	10								
3	0	0	0	5	5	5	5	5	5	5	13	13	13								
4	0	0	0	5	5	5	6	6	6	11	13	13	13								

$x = 20$

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

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***j* (total weight)**

***i* (number of items seen)**

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5	5							
2	0	0	0	5	5	5	5	5	5	5	5	10	10	10							
3	0	0	0	5	5	5	5	5	5	5	13	13	13								
4	0	0	0	5	5	5	6	6	6	11	13	13	13								

Example: The 0-1 Knapsack Problem

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v:	\$5	\$5	\$13	\$6

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***j* (total weight)**

***i* (number of items seen)**

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5	5							
2	0	0	0	5	5	5	5	5	5	5	5	10	10	10							
3	0	0	0	5	5	5	5	5	5	5	13	13	13	18							
4	0	0	0	5	5	5	6	6	6	11	13	13	13								

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***j* (total weight)**

***i* (number of items seen)**

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5	5							
2	0	0	0	5	5	5	5	5	5	5	5	10	10	10							
3	0	0	0	5	5	5	5	5	5	5	13	13	13	18							
4	0	0	0	5	5	5	6	6	6	11	13	13	13	18							

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***j* (total weight)**

***i* (number of items seen)**

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5	5	5						
2	0	0	0	5	5	5	5	5	5	5	5	10	10	10	10						
3	0	0	0	5	5	5	5	5	5	5	13	13	13	18	18						
4	0	0	0	5	5	5	6	6	6	11	13	13	13	18	18						

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***i* (number of items seen)**

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5	5	5	5					
2	0	0	0	5	5	5	5	5	5	5	5	10	10	10	10	10					
3	0	0	0	5	5	5	5	5	5	5	13	13	13	18	18	18					
4	0	0	0	5	5	5	6	6	6	11	13	13	13	18	18	18					

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***j* (total weight)**

***i* (number of items seen)**

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5				
2	0	0	0	5	5	5	5	5	5	5	5	10	10	10	10	10	10				
3	0	0	0	5	5	5	5	5	5	5	13	13	13	18	18	18	18				
4	0	0	0	5	5	5	6	6	6	11	13	13	13	18	18	18					

$x = 20$

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j **(total weight)**

i (number of items seen)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5				
2	0	0	0	5	5	5	5	5	5	5	5	10	10	10	10	10	10				
3	0	0	0	5	5	5	5	5	5	5	13	13	13	18	18	18	18				
4	0	0	0	5	5	5	6	6	6	11	13	13	13	18	18	18	19				

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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
2	0	0	0	5	5	5	5	5	5	5	5	10	10	10	10	10	10	10	10	10	10
3	0	0	0	5	5	5	5	5	5	5	13	13	13	18	18	18	18	18	18	18	18
4	0	0	0	5	5	5	6	6	6	11	13	13	13	18	18	18	19	19	19	19	19

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	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
2	0	0	0	5	5	5	5	5	5	5	5	10	10	10	10	10	10	10	10	10	10
3	0	0	0	5	5	5	5	5	5	5	13	13	13	18	18	18	18	18	18	18	18
4	0	0	0	5	5	5	6	6	6	11	13	13	13	18	18	18	19	19	19	19	19

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	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
2	0	0	0	5	5	5	5	5	5	5	5	10	10	10	10	10	10	10	10	10	10
3	0	0	0	5	5	5	5	5	5	5	13	13	13	18	18	18	18	18	18	18	18
4	0	0	0	5	5	5	6	6	6	11	13	13	13	18	18	18	19	19	19	19	19

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***j* (total weight)**

***i* (number of items seen)**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
2	0	0	0	5	5	5	5	5	5	5	5	10	10	10	10	10	10	10	10	10	10
3	0	0	0	5	5	5	5	5	5	5	13	13	13	18	18	18	18	18	18	18	18
4	0	0	0	5	5	5	6	6	6	11	13	13	13	18	18	18	19	19	19	19	19

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	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
2	0	0	0	5	5	5	5	5	5	5	5	10	10	10	10	10	10	10	10	10	10
3	0	0	0	5	5	5	5	5	5	5	13	13	13	18	18	18	18	18	18	18	18
4	0	0	0	5	5	5	6	6	6	11	13	13	13	18	18	18	19	19	19	19	19

$x = 20$

Example: The 0-1 Knapsack Problem

w:	3	8	10	6
v:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1, j)\} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \end{cases}$$

j **(total weight)**

***i* (number of items seen)**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
2	0	0	0	5	5	5	5	5	5	5	5	10	10	10	10	10	10	10	10	10	10
3	0	0	0	5	5	5	5	5	5	5	13	13	13	18	18	18	18	18	18	18	18
4	0	0	0	5	5	5	6	6	6	11	13	13	13	18	18	18	19	19	19	19	19